Noise sensitivity on continuous products: an answer to an old question of J. Feldman

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Abstract

A relation between σ-additivity and linearizability, conjectured by Jacob Feldman in 1971 for continuous products of probability spaces, is established by relating both notions to a recent idea of noise stability/sensitivity.

Introduction

A discrete-time random process with independent values is just a sequence of independent random variables, described by the product of a sequence of probability spaces. What could be its continuous-time counterpart? Non-equivalent approaches were proposed [2, 4, 5], the whole picture being still unclear.

Independent σ-fields are a more convenient language than products of probability spaces. Each approach deals with a family $(\mathcal{F}_A)_{A \in \mathcal{A}}$ of sub-σ-fields $\mathcal{F}_A \subset \mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, P)$, indexed by subsets $A \subset T$ of some "base" set T, belonging to an algebra[1] $\mathcal{A}$; the family satisfies

$$(0.1) \qquad\qquad \mathcal{F}_{A \uplus B} = \mathcal{F}_A \otimes \mathcal{F}_B \,.$$

That is, if $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $\mathcal{F}_A$ and $\mathcal{F}_B$ are independent[2] and, taken together, they generate $\mathcal{F}_{A \cup B}$. Approaches differ in additional conditions on $\mathcal{A}$ and $(\mathcal{F}_A)$. Most restrictive approaches admit (generalized versions of) classical results such as Levy-Khintchine formula and Levy-Ito theorem. Less restrictive approaches (at least, some of them) are not at all pathologic, they arise from quite natural finite models whose scaling limits go beyond the classical theory [3, 5, 6].

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[1]That is, $A \in \mathcal{A} \implies T \setminus A \in \mathcal{A}$ and $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.
[2]It means that $P(E \cap F) = P(E)P(F)$ for all $E \in \mathcal{F}_A$, $F \in \mathcal{F}_B$.

The approach used by Feldman in 1971 [2] requires $\mathcal{A}$ to be the Borel σ-field of a standard Borel space, and $(\mathcal{F}_A)$ to be σ-additive in the sense that[3]

$$(0.2) \qquad\qquad A_n \uparrow A \quad \Longrightarrow \quad F_{A_n} \uparrow \mathcal{F}_A \,.$$

The classical theory holds [2] for every *decomposable process,* defined as a family $(X_A)_{A \in \mathcal{A}}$ of random variables[4] X_A such that

$$(0.3) \qquad \begin{aligned} X_A \text{ is } \mathcal{F}_A\text{-measurable}\,, \\ X_{A \uplus B} = X_A + X_B\,, \\ A_n \uparrow A \quad \Longrightarrow \quad X_{A_n} \to X_A\,. \end{aligned}$$

The problem formulated by Feldman [2, Problem 1.9]: (a) Does every $(\mathcal{F}_A)$ possess a nontrivial decomposable process? More strongly: (b) Is every $(\mathcal{F}_A)$ *linearizable,* that is, generated by its decomposable processes?[5] Both questions are answered below in the positive. To this end, a concept of noise stability/sensitivity [1] will be adapted to the continuous case.

Feldman's framework is quite restrictive in demanding $\mathcal{A}$ to be a σ-field. Recent examples [3, 5, 6] provide $\mathcal{F}_A$ only for elementary sets A, that is, finite unions of intervals. (Extending $(\mathcal{F}_A)$ to more general A is often impossible, as will be seen.) Restricting ourselves to intervals with rational endpoints we get a *countable* algebra $\mathcal{A}$ of sets, which is a convenient framework, used in Sections 2, 3.

1 Elementary case

In this section the algebra $\mathcal{A}$ is assumed to be finite. Thus, $\mathcal{A}$ corresponds to a finite partition $T = a_1 \uplus \cdots \uplus a_m$, and

$$\mathcal{F}_T = \mathcal{F}_{a_1} \otimes \cdots \otimes \mathcal{F}_{a_m}\,.$$

Each $A \in \mathcal{A}$ is of the form $A = a_{k_1} \uplus \cdots \uplus a_{k_n}$, and (0.1) means simply $\mathcal{F}_A = \mathcal{F}_{a_{k_1}} \otimes \cdots \otimes \mathcal{F}_{a_{k_n}}$. Ascribing to A the probability[6]

$$(1.1) \qquad \mu_p(A) = \mu_p(a_{k_1} \uplus \cdots \uplus a_{k_n}) = p^n (1-p)^{m-n}$$

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[3]In other words, if $A_1 \subset A_2 \subset \ldots$ and $A = A_1 \cup A_2 \cup \ldots$ then $\mathcal{F}_A$ is generated by $\mathcal{F}_{A_1} \cup \mathcal{F}_{A_2} \cup \ldots$

[4]A random variable is treated as an equivalence class of measurable functions on Ω.

[5]Feldman treats a decomposable process more generally (it is defined on some ideal, not the whole $\mathcal{A}$). We do not need it, since $(\mathcal{F}_A)$ is generated by decomposable processes defined on the whole $\mathcal{A}$.

[6]That is *not* the probability P appearing in $(\Omega, \mathcal{F}, P)$.

we get Bernoulli measure μ_p on $\mathcal{A}$; $p \in [0, 1]$ is its parameter. Note that μ_p is not a measure on $(T, \mathcal{A})$, it is rather a measure on $(\mathcal{A}, 2^{\mathcal{A}})$; in other words, $\mathcal{A}$ is treated here as just a set (not an algebra), equipped with the σ-field $2^{\mathcal{A}}$ of all its subsets.[7]

Imagine that $A, B \in \mathcal{A}$ are chosen at random, independently, according to μ_{p_1} and μ_{p_2} respectively; then $A \cap B$ is a random set distributed $\mu_{p_1 p_2}$. In other words,

$$(1.2) \qquad \mu_{p_1} * \mu_{p_2} = \mu_{p_1 p_2} \,;$$

here the convolution $(*)$ of measures on $\mathcal{A}$ (that is, on $(\mathcal{A}, 2^{\mathcal{A}})$) is taken with respect to the semigroup operation of intersection, $\mathcal{A} \times \mathcal{A} \ni (A, B) \mapsto A \cap B \in \mathcal{A}$. The corresponding continuous-time Markov process on $\mathcal{A}$ (its time t is related to p by $p = e^{-t}$) is easy to describe; initially (at $t = 0$) the random set is the whole T; during an infinitesimal time interval $(t, t + dt)$ each a_k is excluded from the random set with probability dt; choices are independent for $k = 1, \ldots, m$; if a_k was excluded before, nothing happens.

A conditional expectation operator corresponds to every $A \in \mathcal{A}$,

$$(1.3)$$
$$E_A : L_2(\mathcal{F}_T) \to L_2(\mathcal{F}_T), \qquad E_A(X) = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_A \right), \quad E_A = \Pr_{L_2(\mathcal{F}_A)},$$

just the orthogonal projection onto[8] $L_2(\mathcal{F}_A) \subset L_2(\mathcal{F}_T)$. Note that $E_T = \mathbf{1}$ (since the operators act on $L_2(\mathcal{F}_T)$, not the whole $L_2(\mathcal{F})$), and

$$(1.4) \qquad E_A E_B = E_{A \cap B} \,,$$

however, $E_{A \uplus B}$ is not $E_A + E_B$; $(E_A)_{A \in \mathcal{A}}$ is not a projection measure on $(T, \mathcal{A})$. In order to get a joint diagonalization of the commuting operators E_A, introduce for every $A \in \mathcal{A}$ a space H_A consisting of all $X \in L_2(\mathcal{F}_A)$ that are orthogonal to $L_2(\mathcal{F}_B)$ for all $B \subset A$, $B \neq A$. We have

$$L_2(\mathcal{F}_T) = \bigoplus_{A \in \mathcal{A}} H_A \,;$$

$$(1.5)$$

$$X = \sum_{A \in \mathcal{A}} X_A, \quad X_A \in H_A \quad \Longrightarrow \quad E_A X = \sum_{B \subset A} X_B \,.$$

Note that $H_\emptyset = L_2(\mathcal{F}_\emptyset)$ is the one-dimensional space of constants, and

$$(1.6) \qquad H_{A \uplus B} = H_A \otimes H_B$$

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[7]Thus, $\mu_p(A)$ should be written rigorously as $\mu_p(\{A\})$.
[8]$L_2(\mathcal{F}_A)$ means $L_2(\Omega, \mathcal{F}_A, P)$.

in the sense that, for any two disjoint $A, B \in \mathcal{A}$, random variables of the form XY for $X \in H_A$, $Y \in H_B$ (belong to and) span $H_{A \cup B}$. In other words,

$$(1.7) \qquad H_{a_{k_1} \uplus \cdots \uplus a_{k_n}} = H_{a_{k_1}} \otimes \cdots \otimes H_{a_{k_n}} ;$$

a proof for $H_{a_1 \cup a_2}$ (general case being similar) consists in choosing orthogonal bases $(X_i)_i$ in $L_2(\mathcal{F}_{a_1})$, $X_0 = \mathbf{1}$, and $(Y_j)_j$ in $L_2(\mathcal{F}_{a_2})$, $Y_0 = \mathbf{1}$, and considering the basis $(X_i Y_j)_{i,j}$ in $L_2(\mathcal{F}_{a_1}) \otimes L_2(\mathcal{F}_{a_2}) = L_2(\mathcal{F}_{a_1} \otimes \mathcal{F}_{a_2}) = L_2(\mathcal{F}_{a_1 \cup a_2})$. So,

$$
\begin{aligned}
L_2(\mathcal{F}_T) &= L_2(\mathcal{F}_{a_1}) \otimes \cdots \otimes L_2(\mathcal{F}_{a_m}) = \\
&= \left(H_\emptyset \oplus H_{a_1} \right) \otimes \cdots \otimes \left(H_\emptyset \oplus H_{a_m} \right) = \bigoplus_{A \in \mathcal{A}} H_A ,
\end{aligned}
$$

$$(1.8)$$

$$
\mathrm{Pr}_{H_A} = \left(\bigotimes_{a \subset A} \mathrm{Pr}_{H_a} \right) \otimes \left(\bigotimes_{a \subset T \setminus A} \left(\mathbf{1} - \mathrm{Pr}_{H_a} \right) \right) .
$$

Combining the conditional expectations with the convolution semigroup, we get an operator semigroup

$$U_t : L_2(\mathcal{F}_T) \to L_2(\mathcal{F}_T) ,$$

$$(1.9) \qquad U_t = \int E_A \, d\mu_p(A) = \sum_{A \in \mathcal{A}} \mu_p(A) E_A \quad \text{where } p = e^{-t} ,$$

$$U_s U_t = U_{s+t} , \qquad U_0 = \mathbf{1} .$$

In the language of tensor products,

$$(1.10) \quad U_t = \underbrace{\left(\mathbf{1} \oplus e^{-t} \cdot \mathbf{1} \right)}_{\text{on } H_\emptyset \oplus H_{a_1}} \otimes \cdots \otimes \underbrace{\left(\mathbf{1} \oplus e^{-t} \cdot \mathbf{1} \right)}_{\text{on } H_\emptyset \oplus H_{a_m}} =$$

$$= \sum_{A \in \mathcal{A}} \left(\bigotimes_{a \subset A} e^{-t} \cdot \mathbf{1} \right) \otimes \left(\bigotimes_{a \subset T \setminus A} \mathbf{1} \right) ,$$

and we get eigenspaces

$$
\begin{aligned}
H_n &= \bigoplus \{ H_A : A = a_{k_1} \uplus \cdots \uplus a_{k_n}, \ k_1 < \cdots < k_n \} ; \\
H_0 &= H_\emptyset = L_2(\mathcal{F}_\emptyset) = \text{constants} , \quad H_m = H_T = H_{a_1} \otimes \cdots \otimes H_{a_m} , \\
&\qquad L_2(\mathcal{F}_T) = H_0 \oplus \cdots \oplus H_m ; \\
X &\in H_n \quad \Longrightarrow \quad U_t X = e^{-nt} X , \\
&\qquad \text{spec } U_t = \{ 1, e^{-t}, e^{-2t}, \ldots, e^{-mt} \} , \\
&\qquad U_\infty = \mathbb{E}(\cdot) ;
\end{aligned}
$$

$$(1.11)$$

the latter means that $\lim_{t\to\infty} U_t X = \mathbb{E}(X) \cdot \mathbf{1}$. According to 1.10, $U_t^A : L_2(\mathcal{F}_A) \to L_2(\mathcal{F}_A)$ for $A \in \mathcal{A}$ may be defined naturally, giving

$$U_t^{A \uplus B} = U_t^A \otimes U_t^B, \qquad U_t^{\emptyset} = \mathbf{1}, \quad U_t^T = U_t,$$

$$U_\infty^A \otimes U_0^{T \setminus A} = \mathbb{E}\left(\cdot \,\big|\, \mathcal{F}_{T \setminus A}\right),$$

(1.12)
$$\mathrm{Pr}_{H_A} = \left(\bigotimes_{a \subset A}(\mathbf{1} - U_\infty^a)\right) \otimes \Big(\underbrace{\bigotimes_{a \subset T \setminus A} U_\infty^a}_{=U_\infty^{T \setminus A}} \Big).$$

Introduce generators:

$$U_t = \exp(-t\mathbf{N}), \quad \mathrm{spec}(\mathbf{N}) = \{0, 1, 2, \ldots, m\},$$
$$X \in H_n \implies \mathbf{N}X = nX;$$

(1.13)
$$U_t^A = \exp(-t\mathbf{N}^A), \quad \mathbf{N}^A : L_2(\mathcal{F}_A) \to L_2(\mathcal{F}_A),$$
$$\mathbf{N}^T = \mathbf{N}; \qquad \mathbf{N}_{a_k} X = X - \mathbb{E}X \quad \text{for } X \in L_2(\mathcal{F}_{a_k});$$
$$\mathbf{N}^{A \uplus B} = \mathbf{N}^A \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{N}^B.$$

The probabilistic meaning of U_t may be described roughly by saying that each of our m pieces of data is unreliable, it is either correct (with probability p) or totally wrong (with probability $1 - p$). More exactly: any random variable $X \in L_2(\mathcal{F}_T)$ is a function, $X = \varphi(Y_1, \ldots, Y_m)$, of m random variables $Y_1, \ldots, Y_m$ such that Y_k is $\mathcal{F}_{a_k}$-measurable (therefore $Y_1, \ldots, Y_m$ are independent). Introduce independent copies $Z_1, \ldots, Z_m$ of $Y_1, \ldots, Y_m$, and a random set $A \in \mathcal{A}$ distributed μ_p and independent of $Y_1, \ldots, Y_m$, $Z_1, \ldots, Z_m$. Define $Y_1', \ldots, Y_m'$ as follows: if $a_k \subset A$ then $Y_k' = Y_k$, otherwise $Y_k' = Z_k$. We have

(1.14)
$$\mathbb{E}\left(\varphi(Y_1, \ldots, Y_m)\,\big|\, Y_1', \ldots, Y_m'\right) = \psi(Y_1', \ldots, Y_m'),$$
$$U_t\left(\varphi(Y_1, \ldots, Y_m)\right) = \psi(Y_1, \ldots, Y_m),$$

which follows by averaging in A of

$$\mathbb{E}\left(\varphi(Y_1, \ldots, Y_m)\,\big|\, A; Y_1', \ldots, Y_m'\right) = \psi_A(Y_1', \ldots, Y_m'),$$
$$E_A\left(\varphi(Y_1, \ldots, Y_m)\right) = \psi_A(Y_1, \ldots, Y_m).$$

The reader may also imagine the corresponding continuous-time Markov process; when a_k is excluded from our random set, the k-th portion of data is immediately replaced with an independent copy. Such functions as[9] $t \mapsto \|X - U_t X\|$, $t \mapsto \|X\| - \|U_t X\|$, or $t \mapsto \left((\mathbf{1} - U_t)X, X\right)$ may

[9]For their interrelations see the proof of Lemma 2.5.

be used for describing noise sensitivity of a random variable X. The more the functions, the more sensitive is X. Least sensitive (most stable) are elements of $H_1 = H_{a_1} \oplus \cdots \oplus H_{a_m}$, that is, random variables of the form $X = X_1 + \cdots + X_m$, where $X_k \in H_{a_k}$ (which means $X_k \in L_2(\mathcal{F}_{a_k})$, $\mathbb{E}X_k = 0$); these satisfy $U_t X = e^{-t} X$. Most sensitive are elements of $H_m = H_T$, that is, linear combinations of random variables of the form $X = X_1 \ldots X_m$ (X_k being as above); these satisfy $U_t X = e^{-mt} X$. The concept of noise sensitivity, quantitative for finite $\mathcal{A}$, becomes qualitative for infinite $\mathcal{A}$, as we'll see in the next section.

The following result shows that contractions do not increase sensitivity.

1.15. Lemma. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \le |x - y|$ for all x, y. Then[10]

$$\big((\mathbf{1} - U_t) f(X), f(X)\big) \le \big((\mathbf{1} - U_t) X, X\big)$$

for all $t \in [0, \infty)$ and $X \in L_2(\mathcal{F}_T)$.

Proof. Introduce $Y_1, \ldots, Y_m$, $Y_1', \ldots, Y_m'$, and φ, ψ as in (1.14); note that $Y_1', \ldots, Y_m'$ are independent and distributed like $Y_1, \ldots, Y_m$; we have

$$\mathbb{E}\big(\varphi(Y_1, \ldots, Y_m) - \varphi(Y_1', \ldots, Y_m')\big)^2 =$$
$$\mathbb{E}\big(\varphi(Y_1, \ldots, Y_m)\big)^2 + \mathbb{E}\big(\varphi(Y_1', \ldots, Y_m')\big)^2 - 2\mathbb{E}\big(\varphi(Y_1, \ldots, Y_m)\varphi(Y_1', \ldots, Y_m')\big) =$$
$$= \|X\|^2 + \|X\|^2 - 2\mathbb{E}\big(\varphi(Y_1', \ldots, Y_m')\mathbb{E}\big(\varphi(Y_1, \ldots, Y_m)\,\big|\,Y_1', \ldots, Y_m'\big)\big) =$$
$$= 2\|X\|^2 - 2\mathbb{E}\big(\varphi(Y_1', \ldots, Y_m')\psi(Y_1', \ldots, Y_m')\big) =$$
$$= 2\|X\|^2 - 2\big(U_t X, X\big) = 2\big((\mathbf{1} - U_t) X, X\big),$$

as well as

$$\mathbb{E}\big(f(\varphi(Y_1, \ldots, Y_m)) - f(\varphi(Y_1', \ldots, Y_m'))\big)^2 = 2\big((\mathbf{1} - U_t) f(X), f(X)\big).$$

However,
$$|f(\varphi(Y_1, \ldots, Y_m)) - f(\varphi(Y_1', \ldots, Y_m'))| \le |\varphi(Y_1, \ldots, Y_m) - \varphi(Y_1', \ldots, Y_m')|. \quad \square$$

Similarly, if $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $|f(x_1, x_2) - f(y_1, y_2)| \le \big((x_1 - y_1)^2 + (x_2 - y_2)^2\big)^{1/2}$ then

$$(1.16) \quad \big((\mathbf{1} - U_t) f(X, Y),\, f(X, Y)\big) \le \big((\mathbf{1} - U_t) X,\, X\big) + \big((\mathbf{1} - U_t) Y,\, Y\big)$$

for all $X, Y \in L_2(\mathcal{F}_T)$ and $t \in [0, \infty)$. The same for $f : \mathbb{R}^d \to \mathbb{R}$.

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[10]For a random variable $X : \Omega \to \mathbb{R}$, $f(X)$ denotes the composition $f \circ X : \Omega \xrightarrow{X} \mathbb{R} \xrightarrow{f} \mathbb{R}$.

2 Stability, sensitivity, linearizability

In this section the algebra $\mathcal{A}$ is assumed to be countable. For example, it may be the algebra generated by intervals $(r, s) \subset \mathbb{R}$ with rational r, s, or the algebra of all cylindrical subsets of $\{0, 1\}^{\mathbb{Z}}$. Being countable, $\mathcal{A}$ is the union of a sequence of its finite subalgebras:

(2.1) $\quad \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots, \qquad \mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ are finite subalgebras of $\mathcal{A}$.

The freedom in choosing the sequence $(\mathcal{A}_m)$ is of no importance for us due to the following "cofinality argument". Let φ be a function defined on the set of all finite subalgebras of $\mathcal{A}$ and such that $\lim_m \varphi(\mathcal{A}_m)$ exists for every sequence $(\mathcal{A}_m)$ satisfying (2.1). Then the limit is the same for all such sequences. Proof: if $(\mathcal{A}_m)$ and $(\mathcal{A}'_m)$ are two such sequences, then we can choose $m_1 < m_2 < \dots$ and $m'_1 < m'_2 < \dots$ such that

(2.2) $$\mathcal{A}_{m_1} \subset \mathcal{A}'_{m'_1} \subset \mathcal{A}_{m_2} \subset \mathcal{A}'_{m'_2} \subset \dots,$$

therefore the sequence $\varphi(\mathcal{A}_{m_1}), \varphi(\mathcal{A}'_{m'_1}), \varphi(\mathcal{A}_{m_2}), \varphi(\mathcal{A}'_{m'_2}), \dots$ must have a limit.

As before, $(\mathcal{F}_A)_{A \in \mathcal{A}}$ satisfying (0.1) is considered. (No other assumptions, such as (0.2).) Still, conditional expectation operators E_A are defined, see (1.3), (1.4).

Restricting $(\mathcal{F}_A)$ to $A \in \mathcal{A}_m$ we get the elementary case of Sect. 1. Probability measures $\mu_p^{(m)}$ are defined, see (1.1) and (1.2), subspaces $H_A^{(m)}$ for $A \in \mathcal{A}_m$, see (1.5),[11] operator semigroups $U_t^{(m)}$, see (1.9), their eigenspaces $H_n^{(m)}$, see (1.11), and generators $\mathbf{N}_m$, see (1.13). All $U_t^{(m)}$ belong to the commutative algebra generated by operators of conditional expectation E_A, $A \in \mathcal{A}$. Compare $H_A^{(m)}$ and $H_B^{(m+1)}$ for $A \in \mathcal{A}_m$, $B \in \mathcal{A}_{m+1}$; the second space is either included into the first, or orthogonal to it; namely, if A is the least element of $\mathcal{A}_m$ containing B ($\mathcal{A}_m$-saturation of B), then $H_B^{(m+1)} \subset H_A^{(m)}$, otherwise $H_B^{(m+1)} \perp H_A^{(m)}$. If $X \in H_B^{(m+1)} \subset H_A^{(m)}$ then $U_t^{(m)} X = e^{-kt} X$, $U_t^{(m+1)} X = e^{-lt} X$ with $k \leq l$ (since saturation does not increase the number of atoms). So,

(2.3) $$\mathbf{N}_m \leq \mathbf{N}_{m+1}; \qquad U_t^{(m)} \geq U_t^{(m+1)}.$$

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[11]The reader may guess that the decomposition of $L_2(\mathcal{F}_T)$ into the direct sum of $H_A^{(m)}$, $A \in \mathcal{A}_m$, has a kind of limit for $m \to \infty$. That is true; the limit is described in [4, Sect. 2] in terms of direct integrals of Hilbert spaces (for somewhat more restrictive framework, though). In the present paper, direct integrals do not appear explicitly; however, most of the text is in fact translated from that language.

It follows easily that the limit exists,

$$U_t = \lim_{m \to \infty} U_t^{(m)} \quad \text{in the sense that } \forall X \in L_2 \; \|U_t^{(m)} X - U_t X\| \to 0;$$
(2.4)
$$U_s U_t = U_{s+t}; \qquad \|U_t\| \le 1.$$

The limit, U_t, does not depend on the choice of $(\mathcal{A}_m)$ due to the cofinality argument (see (2.2)). Also, U_t commute with all E_A.

The limit of generators, $\lim_m \mathbf{N}_m$, need not exist; $\|\mathbf{N}_m X\|$ can tend to ∞ for some X. Accordingly, the operator semigroup (U_t) need not be continuous at $t = 0$.

2.5. Lemma. There exists a sub-σ-field $\mathcal{F}_{\text{stable}} \subset \mathcal{F}_T$ such that

$$X \in L_2(\mathcal{F}_{\text{stable}}) \quad \Longleftrightarrow \quad \|X - U_t X\| \xrightarrow[t \to 0]{} 0.$$

Proof. First, the following three properties of X are equivalent: (a) $\|X - U_t X\| \xrightarrow[t \to 0]{} 0$; (b) $\|U_t X\| \xrightarrow[t \to 0]{} \|X\|$; (c) $\big((\mathbf{1} - U_t)X, X\big) \xrightarrow[t \to 0]{} 0$. Indeed, (c) $\Longrightarrow$ (a) since $\|X - U_t X\|^2 = \big((\mathbf{1} - U_t)^2 X, X\big) \le \big((\mathbf{1} - U_t)X, X\big)$; (a) $\Longrightarrow$ (b) since $\|U_t X\| \ge \|X\| - \|X - U_t X\|$; (b) $\Longrightarrow$ (c) since $\big((\mathbf{1} - U_t)X, X\big) = \|X\|^2 - \|U_{t/2}X\|^2$.

The set $H_{\text{stable}} = \{X \in L_2(\mathcal{F}_T) : \|X - U_t X\| \xrightarrow[t \to 0]{} 0\} = \{X \in L_2(\mathcal{F}_T) : \big((\mathbf{1} - U_t)X, X\big) \xrightarrow[t \to 0]{} 0\}$ is a closed linear subspace of L_2. By Lemma 1.15, if $X, Y \in H_{\text{stable}}$ then $\min(X, Y), \max(X, Y) \in H_{\text{stable}}$. Also, H_{stable} contains constants. It is well-known that such a space is the whole $L_2(\mathcal{F}_{\text{stable}})$ where $\mathcal{F}_{\text{stable}}$ is the σ-field generated by H_{stable}. $\qquad \square$

We have

(2.6)
$$U_t\big(L_2(\mathcal{F}_{\text{stable}})\big) \subset L_2(\mathcal{F}_{\text{stable}}) \quad \text{for all } t \in [0, \infty),$$
$$E_A\big(L_2(\mathcal{F}_{\text{stable}})\big) \subset L_2(\mathcal{F}_{\text{stable}}) \quad \text{for all } A \in \mathcal{A},$$

since $\|U_s X - U_t U_s X\| = \|U_s(X - U_t X)\| \le \|X - U_t X\|$ and $\|E_A X - U_t E_A X\| = \|E_A(X - U_t X)\| \le \|X - U_t X\|$. Being restricted to $L_2(\mathcal{F}_{\text{stable}})$, the operator semigroup (U_t) is continuous (in the strong operator topology) and has its generator $\mathbf{N} = \lim_m \mathbf{N}_m$, $\text{spec}\,\mathbf{N} \subset \{0, 1, 2, \dots\}$; denote its eigenspaces by H_n;

(2.7)
$$U_t = e^{-t\mathbf{N}} \quad \text{on } L_2(\mathcal{F}_{\text{stable}}),$$
$$L_2(\mathcal{F}_{\text{stable}}) = H_0 \oplus H_1 \oplus H_2 \oplus \dots,$$
$$U_t = e^{-nt} \quad \text{on } H_n,$$
$$E_A(H_n) \subset H_n \quad \text{for all } A \in \mathcal{A};$$

check the latter: $X \in H_n \implies U_t E_A X = E_A U_t X = e^{-nt} E_A X \implies E_A X \in H_n$. The relation $\mathbf{N}_m \uparrow \mathbf{N}$ implies for all $n \in \{0, 1, 2, \dots\}$

$$(2.8) \qquad H_0 \oplus \cdots \oplus H_n = \bigcap_{m=1}^{\infty} H_0^{(m)} \oplus \cdots \oplus H_n^{(m)}$$

(the intersection of a decreasing sequence of subspaces). Clearly, $H_0 = H_0^{(m)} = H_\emptyset^{(m)}$ is the one-dimensional space of constants.

2.9. Lemma. The following conditions are equivalent for all $X \in L_2(\mathcal{F}_T)$:
  (a) $X \in H_1$;
  (b) $X = E_A X + E_{T \setminus A} X$ for all $A \in \mathcal{A}$;
  (c) $X = E_{A_1} X + \cdots + E_{A_k} X$ for every partition $T = A_1 \uplus \cdots \uplus A_k$ of T into $A_i \in \mathcal{A}$.

Proof. Each element of $H_1^{(m)}$ satisfies (c) for the partition into atoms of $\mathcal{A}_m$. Therefore each element of $H_1 = \cap_m H_1^{(m)}$ satisfies (c) for every partition, and we get (a) $\implies$ (c) $\implies$ (b). For proving (b) $\implies$ (a) assume that X satisfies (b) and prove that $X \in H_1^{(m)}$ for all m. From now on A and B run over $\mathcal{A}_m$. By (1.5), $X = \sum X_A$, $X_A \in H_A^{(m)}$; we have to prove that $X_A = 0$ unless A contains exactly one atom. By (1.5) again,

$$E_A X = \sum_{B \subset A} X_B, \qquad E_{T \setminus A} X = \sum_{B \subset T \setminus A} X_B.$$

We see that $X_\emptyset$ appears twice in $E_A X + E_{T \setminus A} X$, but only once in X, therefore $X_\emptyset = 0$. If B contains at least two atoms, we can choose A such that B intersects both A and $T \setminus A$; then X_B does not appear in $E_A X + E_{T \setminus A} X$, but appears in X, therefore $X_B = 0$. $\qquad\square$

The following is a general fact about Hilbert spaces, irrespective of any probability theory.

2.10. Lemma. Assume that H' and H'' are Hilbert spaces, $H = H' \otimes H''$, and subspaces are given, $H' \supset H_1' \supset H_2' \supset \dots$ and $H'' \supset H_1'' \supset H_2'' \supset \dots$ Then

$$\bigcap_m (H_m' \otimes H_m'') = \left(\bigcap_m H_m' \right) \otimes \left(\bigcap_m H_m'' \right).$$

Proof. Denoting $H_0' = H'$, $H_\infty' = \cap H_m'$, we have

$$H' = H_\infty' \oplus \bigoplus_{m=0}^{\infty} (H_m' \ominus H_{m+1}')$$

9

and the same for H''. Therefore

$$H = \left(H'_\infty \otimes H''_\infty \right) \oplus \left(\bigoplus_{m=0}^{\infty} (H'_m \ominus H'_{m+1}) \otimes H''_\infty \right) \oplus$$

$$\oplus \left(H'_\infty \otimes \bigoplus_{n=0}^{\infty} (H''_n \ominus H''_{n+1}) \right) \oplus \left(\bigoplus_{m,n} (H'_m \ominus H'_{m+1}) \otimes (H''_n \ominus H''_{n+1}) \right).$$

The space $H'_m \otimes H''_m$ contains some of the terms, and is orthogonal to others. Only the term $H'_\infty \otimes H''_\infty$ is contained in $H'_m \otimes H''_m$ for all m. $\qquad\square$

2.11. Lemma. For any m and any two different atoms a, b of $\mathcal{A}_m$,

$$H_2 \cap H_{a \cup b}^{(m)} = \left(H_1 \cap H_a^{(m)} \right) \otimes \left(H_1 \cap H_b^{(m)} \right).$$

Proof. $H_{a \cup b}^{(m)} = H_a^{(m)} \otimes H_b^{(m)}$ by (1.6); $H_2 \cap H_{a \cup b}^{(m)} = (H_0 \oplus H_1 \oplus H_2) \cap H_{a \cup b}^{(m)} = \cap_k \left(H_0^{(m+k)} \oplus H_1^{(m+k)} \oplus H_2^{(m+k)} \right) \cap H_{a \cup b}^{(m)} = \cap_k H_2^{(m+k)} \cap H_{a \cup b}^{(m)}$ by (2.8); note that the space decreases when k increases. Similarly, $H_1 \cap H_a^{(m)} = \cap_k H_1^{(m+k)} \cap H_a^{(m)}$ and $H_1 \cap H_b^{(m)} = \cap_k H_1^{(m+k)} \cap H_b^{(m)}$. By Lemma 2.10 it suffices to prove that

$$H_2^{(m+k)} \cap H_{a \cup b}^{(m)} = \left(H_1^{(m+k)} \cap H_a^{(m)} \right) \otimes \left(H_1^{(m+k)} \cap H_b^{(m)} \right)$$

for $k = 1, 2, \ldots$ However, $H_2^{(m+k)}$ is (by definition) the direct sum of $H_{c \cup d}^{(m+k)}$ over atoms c, d of $\mathcal{A}_{m+k}$, $c \neq d$, and $H_2^{(m+k)} \cap H_{a \cup b}^{(m)}$ is such a sum over $c \subset a$, $d \subset b$. It remains to note that $H_{c \cup d}^{(m+k)} = H_c^{(m+k)} \otimes H_d^{(m+k)}$. $\qquad\square$

2.12. Theorem. The σ-field generated by H_1 is equal to $\mathcal{F}_{\text{stable}}$.

Proof. Denote by $\mathcal{F}_n$ the σ-field generated by H_n. It suffices to prove that $\mathcal{F}_n \subset \mathcal{F}_1$ for all n, since $L_2(\mathcal{F}_{\text{stable}}) = H_0 \oplus H_1 \oplus H_2 \oplus \ldots$ (see (2.7)). I give a proof for $n = 2$; it has a straightforward generalization for higher n.

We have to prove that $H_2 \subset L_2(\mathcal{F}_1)$. For each m, $H_2 = \left(H_2 \cap H_1^{(m)} \right) \oplus \left(H_2 \cap H_2^{(m)} \right)$ (since H_2 is invariant under all E_A). However, $H_1^{(m)}$ decreases to H_1, and H_1 is orthogonal to H_2. Therefore the union of $H_2 \cap H_2^{(m)}$ is dense in H_2; it remains to prove that $H_2 \cap H_2^{(m)} \subset L_2(\mathcal{F}_1)$ for all m. Note that $H_2 \cap H_2^{(m)}$ is the direct sum of $H_2 \cap H_{a \cup b}^{(m)}$ over atoms a, b of $\mathcal{A}_m$, $a \neq b$ (since, again, H_2 is invariant under all E_A). Lemma 2.11 reduces the needed inclusion to an evident fact, $\left(H_1 \cap H_a^{(m)} \right) \otimes \left(H_1 \cap H_b^{(m)} \right) \subset L_2(\mathcal{F}_1)$. $\qquad\square$

A canonical isomorphism between H_n and $\underbrace{H_1 \otimes \cdots \otimes H_1}_{n}$ is given by Wick products,[12] but is not needed here.

2.13. Definition. (a) A random variable $X \in L_2(\mathcal{F}_T)$ is called *stable*, if $X \in L_2(\mathcal{F}_{\text{stable}})$, and *sensitive*, if $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}}\right) = 0$.
(b) Equivalently, a random variable $X \in L_2(\mathcal{F}_T)$ is called *stable*, if $\|X - U_t X\| \xrightarrow[t \to 0]{} 0$, and *sensitive*, if $U_t X = 0$ for all $t \in (0, \infty)$.

The two definitions of stability are equivalent evidently (recall 2.5), of sensitivity — due to the following result.

2.14. Lemma. The following conditions are equivalent for all $X \in L_2(\mathcal{F}_T)$:
(a) $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}}\right) = 0$;
(b) $U_t X = 0$ for all $t > 0$.

Proof. (b) $\implies$ (a): Let $Y \in L_2(\mathcal{F}_{\text{stable}})$, then $(X, Y) = \lim_{t \to 0}(X, U_t Y) = \lim_{t \to 0}(U_t X, Y) = 0$ by (b).
(a) $\implies$ (b): It suffices to prove that $\|U_t X\| \leq e^{-nt}\|X\|$ for all $X \in L_2(\mathcal{F}_T)$ orthogonal to $H_0 \oplus \cdots \oplus H_{n-1}$. By (2.8) we may assume that X is orthogonal to $H_0^{(m)} \oplus \cdots \oplus H_{n-1}^{(m)}$ for some m (since such vectors are dense in $L_2(\mathcal{F}_T) \ominus (H_0 \oplus \cdots \oplus H_{n-1})$). For such X, $\|U_t X\| \leq \|U_t^{(m)} X\| \leq e^{-nt}\|X\|$. $\square$

So, in terms of $U_{0+} X = \lim_{t \to 0, t > 0} U_t X$ we have

$$
\begin{aligned}
L_2(\mathcal{F}_T) &= \{X : X \text{ is stable}\} \oplus \{X : X \text{ is sensitive}\}, \\
X \text{ is stable} &\iff U_{0+} X = X, \\
X \text{ is sensitive} &\iff U_{0+} X = 0; \\
\mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{\text{stable}}\right) &= U_{0+}.
\end{aligned}
$$
(2.15)

Similarly, for any $A \in \mathcal{A}$

$$
\mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{\text{stable}}^A\right) = U_{0+}^A \quad \text{on } L_2(\mathcal{F}_A)
$$
(2.16)

for some $\mathcal{F}_{\text{stable}}^A$, and $\mathcal{F}_{\text{stable}}^A = \mathcal{F}_A \cap \mathcal{F}_{\text{stable}}$, since $U_t^A X = U_t X$ for $X \in L_2(\mathcal{F}_A)$; also, $\mathcal{F}_{\text{stable}}^A$ is generated by $H_1^A = H_1 \cap L_2(\mathcal{F}_A)$, therefore

$$
\mathcal{F}_{\text{stable}}^{A \uplus B} = \mathcal{F}_{\text{stable}}^A \otimes \mathcal{F}_{\text{stable}}^B,
$$
(2.17)

---

[12]Given $X, Y \in H_1$, we may define their Wick product,

$$
:XY: = \lim_{m \to \infty} \sum_{a \neq b} (E_a X)(E_b Y).
$$

The sum is taken over all unordered pairs $\{a, b\}$ of different atoms of $\mathcal{A}_m$. The same for $:XYZ:$ and so on.

which means that $\left(\mathcal{F}_{\text{stable}}^A\right)_{A\in\mathcal{A}}$ is another family satisfying (0.1), the *stable (or linearizable) part* of the given family $(\mathcal{F}_A)_{A\in\mathcal{A}}$. (See also [5, Th. 1.7].)

3 Stability and extendibility

We still work with a countable algebra $\mathcal{A}$ and a family $(\mathcal{F}_A)_{A\in\mathcal{A}}$ satisfying (0.1). Striving to extend the family from the algebra $\mathcal{A}$ to the σ-field generated by $\mathcal{A}$ we can face the following obstacle.

Let $A_k \in \mathcal{A}$, $A_1 \subset A_2 \subset \dots$; consider two σ-fields: $\bigvee \mathcal{F}_{A_k}$ (the least σ-field containing all $\mathcal{F}_{A_k}$), and $\bigwedge \mathcal{F}_{T\setminus A_k} = \bigcap \mathcal{F}_{T\setminus A_k}$ (the intersection of all $\mathcal{F}_{T\setminus A_k}$). It is easy to see that the two σ-fields are independent. The question is, whether

$$(3.1) \qquad \left(\bigvee_k \mathcal{F}_{A_k}\right) \otimes \left(\bigwedge_k \mathcal{F}_{T\setminus A_k}\right) = \mathcal{F}_T\,,$$

or not. That is, whether the two σ-fields generate the whole $\mathcal{F}_T$, or not. If they do not, then $(\mathcal{F}_A)$ has no σ-additive (in the sense of (0.2)) extension to a σ-field.

3.2. Theorem. If (3.1) is satisfied for every increasing sequence (A_k), then $\mathcal{F}_T = \mathcal{F}_{\text{stable}}$.

Postpone the proof. Choose $\mathcal{A}_m$ satisfying (2.1). Choose $p_k \in (0,1)$ such that $\sum(1 - p_k) < 1$, say, $p_k = 1 - 2^{-k-1}$. Recall probability measures $\mu_p^{(m)}$.

3.3. Lemma. There exists a sequence $m_1 < m_2 < \dots$ such that $\left(\mu_{p_1}^{(m_1)} \otimes \mu_{p_2}^{(m_2)} \otimes \dots\right)$-almost all sequences $(A_1, A_2, \dots)$, $A_k \in \mathcal{A}_{m_k}$, satisfy

$$\bigcap_{k=1}^{\infty} \mathcal{F}_{A_k} \subset \mathcal{F}_{\text{stable}}\,.$$

Proof. If $\bigcap \mathcal{F}_{A_k}$ is not contained in $\mathcal{F}_{\text{stable}}$ then there exists $X \in L_2(\bigcap \mathcal{F}_{A_k})$, $X \neq 0$, orthogonal to $L_2(\mathcal{F}_{\text{stable}})$, that is, sensitive. The case is impossible, if $E_{A_k} X \to 0$ for all sensitive X or, equivalently, for a dense set of such X; the more so, if $\sum_k (E_{A_k} X, X) < \infty$ for all these X. By (1.9), $(U_{t_k}^{(m_k)} X, X)$ is the average of $(E_{A_k} X, X)$ over A_k distributed $\mu_{p_k}^{(m_k)}$; here $t_k = -\ln p_k$. It suffices to choose m_k such that $\sum_k (U_{t_k}^{(m_k)} X, X) < \infty$ for a dense set of sensitive X.

For each sensitive X and each $t > 0$, by 2.14, $(U_t^{(m)} X, X) \to 0$ for $m \to \infty$. Therefore $(U_{t_k}^{(m_k)} X, X) \to 0$ for $k \to \infty$, if m_k grow fast enough. Diagonal argument gives a single sequence (m_k) that serves a given sequence of vectors X. It remains to choose a sequence dense among all sensitive vectors. $\square$

Introduce $\tilde{E}_A$ similar to E_A as follows:

$$(3.4) \qquad \begin{aligned} E_A &= \mathbb{E}\left(\cdot \mid \mathcal{F}_A \right) = \mathbf{1}_A \otimes U_\infty^{T\backslash A} \, ; \\ \tilde{E}_A &= \mathbb{E}\left(\cdot \mid \mathcal{F}_A \vee \mathcal{F}_{\text{stable}} \right) = \mathbf{1}_A \otimes U_{0+}^{T\backslash A} \, ; \end{aligned}$$

here $\mathcal{F}_A \vee \mathcal{F}_{\text{stable}}$ is the σ-field generated by these two σ-fields, and $\mathbf{1}_A$ is the unit operator on $L_2(\mathcal{F}_A)$; the equality $\mathbb{E}\left(\cdot \mid \mathcal{F}_A \vee \mathcal{F}_{\text{stable}} \right) = \mathbf{1}_A \otimes U_{0+}^{T\backslash A}$ follows from (2.16), since by (2.17), $\mathcal{F}_A \vee \mathcal{F}_{\text{stable}} = \mathcal{F}_A \vee \left(\mathcal{F}_{\text{stable}}^A \otimes \mathcal{F}_{\text{stable}}^{T\backslash A} \right) = \mathcal{F}_A \otimes \mathcal{F}_{\text{stable}}^{T\backslash A}$.

We could proceed to $\tilde{U}_t^{(m)}$ similar to $U_t^{(m)}$,

$$(3.5) \qquad \begin{aligned} U_t^{(m)} &= \int E_A \, d\mu_p^{(m)}(A) \, , \\ \tilde{U}_t^{(m)} &= \int \tilde{E}_A \, d\mu_p^{(m)}(A) \, , \end{aligned} \qquad (p = e^{-t})$$

and to $\tilde{U}_t = \lim_m U_t^{(m)}$; however, we need a bit more general construction,

$$(3.6) \qquad \tilde{U}_\mu = \int \tilde{E}_A \, d\mu(A)$$

for an arbitrary probability distribution μ on A (A is treated here as just a countable set). In fact, we need only μ concentrated on a finite set, which is elementary in the sense of Sect. 1.

3.7. Lemma. $\tilde{U}_\mu \leq (1-p)U_{0+} + p \cdot \mathbf{1}$, where[13]

$$p = \sup_{t \in T} \mu\left(\{ A \in \mathcal{A} : A \ni t \} \right) .$$

Proof. Similarly to (1.8), (1.12), for every m,

$$(3.8) \qquad \begin{aligned} L_2(\mathcal{F}_T) &= \bigotimes_a L_2(\mathcal{F}_a) = \bigotimes_a \left(H_{\text{stable}}^a \oplus H_{\text{sensitive}}^a \right) = \bigoplus_{A \in \mathcal{A}_m} \tilde{H}_A \, , \\ \text{Pr}_{\tilde{H}_A} &= \left(\bigotimes_{a \subset A} \underbrace{\text{Pr}_{H_{\text{sensitive}}^a}}_{\mathbf{1} - U_{0+}^a} \right) \otimes \left(\bigotimes_{a \subset T\backslash A} \underbrace{\text{Pr}_{H_{\text{stable}}^a}}_{U_{0+}^a} \right) , \end{aligned}$$

---

[13]Recall that $\mathcal{A}$ is an algebra of subsets of some set T. The latter was mentioned only once, before (0.1), and may be readily avoided now;

$$p = \sup_{B \in \mathcal{A}, B \neq \emptyset} \mu\left(\{ A \in \mathcal{A} : A \supset B \} \right) .$$

where a runs over atoms of $\mathcal{A}_m$, and H^a_{stable}, $H^a_{\text{sensitive}}$ are subspaces of stable and sensitive, respectively, elements of $L_2(\mathcal{F}_a)$.

For every $B \in \mathcal{A}_m$, $B \neq \emptyset$, the operator $\tilde{E}_A$ (recall (3.4)) on H_B is the unit (identity) if $B \subset A$, otherwise it vanishes. Assuming $\mu(\mathcal{A}_m) = 1$ we get $\tilde{U}_\mu = \lambda \cdot \mathbf{1}$ on H_B, where $\lambda = \mu(\{A : A \supset B\}) \leq p$. Therefore $\tilde{U}_\mu \leq p \cdot \mathbf{1}$ on H_B, $B \neq \emptyset$ (note that $\tilde{U}_\mu(H_B) \subset H_B$), while on $H_\emptyset$ we have $U_{0+} = \mathbf{1}$; so, $\tilde{U}_\mu \leq (1-p)U_{0+} + p \cdot \mathbf{1}$ provided that $\mu(\mathcal{A}_m) = 1$ for some m. The general case, $\mu(\mathcal{A}_m) \to 1$, will not be used, and I leave it to the reader. $\qquad\square$

Proof of Theorem 3.2. Choose m_k by Lemma 3.3, then $\cap \mathcal{F}_{A_k} \subset \mathcal{F}_{\text{stable}}$ for μ-almost all (A_k); here $\mu = \otimes_k \mu_{p_k}^{(m_k)}$. On the other hand, for every $t \in T$ and every k,

$$\mu(\{(A_k) : t \in T \setminus (A_1 \cap \cdots \cap A_k)\}) \leq$$
$$\leq \mu_{p_1}^{(m_1)}(\{A_1 : t \in T \setminus A_1\}) + \cdots + \mu_{p_k}^{(m_k)}(\{A_k : t \in T \setminus A_k\}) \leq$$
$$\leq \sum_i (1 - p_i) = q < 1 \, ;$$

by Lemma 3.7,

$$\int \tilde{E}_{T \setminus (A_1 \cap \cdots \cap A_k)} \, d\mu \leq (1-q)U_{0+} + q \cdot \mathbf{1} \, ,$$

therefore

$$\int \left\| \mathbb{E}\left(X \,\Big|\, \bigvee_{k=1}^{\infty} \mathcal{F}_{T \setminus (A_1 \cap \cdots \cap A_k)} \vee \mathcal{F}_{\text{stable}} \right) \right\|^2 d\mu =$$
$$= \lim_{k \to \infty} \int \| \mathbb{E}\left(X \,\big|\, \mathcal{F}_{T \setminus (A_1 \cap \cdots \cap A_k)} \vee \mathcal{F}_{\text{stable}} \right) \|^2 \, d\mu \leq q \|X\|^2$$

for all sensitive $X \in L_2(\mathcal{F}_T)$. Applying (3.1) to the increasing sequence $T \setminus (A_1 \cap \cdots \cap A_k)$ we get

$$\left(\bigvee_{k=1}^{\infty} \mathcal{F}_{T \setminus (A_1 \cap \cdots \cap A_k)} \right) \otimes \left(\bigwedge_{k=1}^{\infty} \mathcal{F}_{A_k} \right) = \mathcal{F}_T$$

for all (A_k), therefore

$$\left(\bigvee_{k=1}^{\infty} \mathcal{F}_{T \setminus (A_1 \cap \cdots \cap A_k)} \right) \vee \mathcal{F}_{\text{stable}} = \mathcal{F}_T$$

for μ-almost all (A_k). So, each sensitive X satisfies $\int \|X\|^2 \, d\mu \leq q\|X\|^2$, that is, $\|X\|^2 \leq q\|X\|^2$, which is impossible unless $X = 0$. $\qquad\square$

So, if $\mathcal{F}_T \neq \mathcal{F}_{\text{stable}}$ then $(\mathcal{F}_A)_{A \in \mathcal{A}}$ has no σ-additive extension. On the other hand, if $\mathcal{F}_T = \mathcal{F}_{\text{stable}}$ then such an extension is usually possible, for a simple reason: E_A restricted to H_1 form a projection-valued finitely additive measure. Conditions well-known in measure theory ensure that a σ-additive extension to a σ-field exists, and we get extended $\mathcal{F}_A$ as generated by extended H_1^A.

A Appendix: The simplest example of sensitivity

> The phenomenon ... tripped up even
> Kolmogorov and Wiener. [7, p. 48]

Two examples of a countable algebra $\mathcal{A}$, mentioned in the beginning of Sect. 2, are nonatomic; corresponding families $(\mathcal{F}_A)_{A \in \mathcal{A}}$ are in general as complicated as continuous-time random processes. The simplest infinite $\mathcal{A}$ consists of all finite and cofinite[14] subsets of $T = \{1, 2, \dots\}$. From now on, $\mathcal{A}$ stands for that algebra; it is purely atomic, and corresponding $(\mathcal{F}_A)_{A \in \mathcal{A}}$ are as simple as discrete-time random processes, that is, random sequences. Not too simple, as we'll see soon ...

Choose some $p \in \{3, 5, 7, 9, \dots\}$ and consider the simple stationary random walk on the finite group $\mathbb{Z}_p$. That is, Ω is the set of all sequences $\omega = (x_0, x_1, x_2, \dots)$, $x_k \in \mathbb{Z}_p$, $x_{k+1} - x_k = \pm 1$; $\mathcal{F}$ is the σ-field generated by cylinder sets $E_{y_0, \dots, y_m} = \{\omega \in \Omega : X_0(\omega) = y_0, \dots, X_m(\omega) = y_m\}$, where $X_k(x_0, x_1, \dots) = x_k$; and P is defined by $P(E_{y_0, \dots, y_m} = p^{-1} 2^{-m}$ whenever $y_k \in \mathbb{Z}_p$, $y_{k+1} - y_k = \pm 1$. So, each of the $\mathbb{Z}_p$-valued random variables $X_0, X_1, \dots$ is uniformly distributed; increments $X_1 - X_0, X_2 - X_1, \dots$ are independent, ± 1 with probabilities $1/2, 1/2$; and the random variables $X_0; X_1 - X_0, X_2 - X_1, \dots$ are independent.

Define σ-fields $\mathcal{F}_A$ for $A \in \mathcal{A}$:

(A.1)
$$\mathcal{F}_{\{k\}} = \sigma(X_k - X_{k-1}),$$
$$\mathcal{F}_{\{k,k+1,\dots\}} = \sigma(X_{k-1}, X_k, X_{k+1}, \dots),$$
$$\mathcal{F}_{\{k_1,\dots,k_n\}} = \mathcal{F}_{k_1} \vee \cdots \vee \mathcal{F}_{k_n},$$
$$\mathcal{F}_{\{k_1,\dots,k_n\} \cup \{k,k+1,\dots\}} = \mathcal{F}_{\{k_1,\dots,k_n\}} \vee \mathcal{F}_{\{k,k+1,\dots\}};$$

here $n \in \{0, 1, \dots\}$, $k, k_1, \dots, k_n \in \{1, 2, \dots\}$, $k_1 < \cdots < k_n < k$, and $\sigma(\dots)$ means the σ-field generated by given random variables. It is not immediately clear that the definition is correct and (0.1) is satisfied, but it is true; you

---

[14]A set is called cofinite if its complement is finite.

may check it, starting with

$$\mathcal{F}_{\{1,\ldots,k-1\}} \otimes \mathcal{F}_{\{k,k+1,\ldots\}} = \mathcal{F}.$$

Condition (3.1) is violated for $A_k = \{1,\ldots,k\}$, since the σ-field $\bigwedge_{k=1}^{\infty} \mathcal{F}_{\{k+1,k+2,\ldots\}}$ is degenerate, while the σ-field $\bigvee_{k=1}^{\infty} \mathcal{F}_{\{1,\ldots,k\}} = \sigma(X_1 - X_0, X_2 - X_1, \ldots)$ contains only sets invariant under the symmetry

(A.2) $R : \Omega \to \Omega, \qquad R(x_0, x_1, \ldots) = (x_0 + 1, x_1 + 1, \ldots).$

(Note that X_0 is not invariant under R.) Therefore $\mathcal{F}_A$ cannot be defined for all $A \subset \{1, 2, \ldots\}$ obeying (0.1), (0.2) and (A.1).

We choose finite subalgebras $\mathcal{A}_m \subset \mathcal{A}$, satisfying (2.1), in a natural way:

atoms of $\mathcal{A}_m$ are $\{1\}, \ldots, \{m-1\}, \text{ and } \{m, m+1, \ldots\}.$

An elementary calculation, starting with

$$X_0 = X_{m-1} - (X_{m-1} - X_{m-2}) - \cdots - (X_1 - X_0),$$

gives

$$U_t^{(m)} \exp\left(\frac{2\pi i}{p} X_0\right) =$$
$$= e^{-t} \exp\left(\frac{2\pi i}{p} X_{m-1}\right) \cdot \prod_{k=1}^{m-1} \left(\cos\frac{2\pi}{p} + ie^{-t}\sin\frac{2\pi}{p}(X_k - X_{k-1})\right);$$
$$\left\|U_t^{(m)} \exp\left(\frac{2\pi i}{p} X_0\right)\right\| = e^{-t}\left(\cos^2\frac{2\pi}{p} + e^{-2t}\sin^2\frac{2\pi}{p}\right)^{(m-1)/2};$$

therefore $U_t \exp\left(\frac{2\pi i}{p} X_0\right) = 0$ for all $t > 0$, which means that

(A.3) $$\exp\left(\frac{2\pi i}{p} X_0\right) \quad \text{is sensitive.}$$

In fact, $\mathcal{F}_{\text{stable}} = \sigma(X_1 - X_0, X_2 - X_1, \ldots)$ is the σ-field of all measurable sets that are invariant under the symmetry R. Accordingly,

(A.4) $$\mathbb{E}\left(X \mid \mathcal{F}_{\text{stable}}\right) = \frac{1}{p}(X + X \circ R + X \circ R^2 + \cdots + X \circ R^{p-1}).$$

Also, it is easy to see that

(A.5) $$H_1 = \{c_1(X_1 - X_0) + c_2(X_2 - X_1) + \ldots : (c_1, c_2, \ldots) \in l_2\}$$

(here $X_k - X_{k-1}$ is treated as taking on values $\pm 1 \in \mathbb{R}$ rather than $\pm 1 \in \mathbb{Z}_p$).

Instead of $\mathbb{Z}_p$ we could consider the unit circle on the complex plane, and some random walk in the circle (or another compact group).

A physicists could write

$$\exp\left(\frac{2\pi i}{p} X_0\right) = \bigotimes_{k=1}^{\infty} \exp\left(\frac{2\pi i}{p}(X_k - X_{k-1})\right)$$

and say: that is just the wave function of an infinite sequence of uncorrelated spins (or quantum bits), all in the same superposition of two basis states. True, the infinite product of independent identically distributed random variables does not converge, but anyway, infinitely many commuting copies of SU(2) act on $L_2(\mathcal{F}_T)$.

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